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Stirling Cryocooler Model with Stratified Cylinders and Quasisteady Heat Exchangers

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A new one-dimensional model of Stirling cryocoolers is presented. A new rate-dependent heat exchanger model is coupled with our existing stratified, isentropic cylinder model. As before, the regenerator is modeled as isothermal, and pressure is modeled as spatially uniform. The heat exchanger model assumes that the time-dependent mass flow rate in the heat exchanger is spatially uniform. This is a fair approximation if the heat exchanger volume is relatively small compared to the displacement. Under that assumption, the conservation laws for mass and energy can be integrated in closed form with respect to the space variable, in all the spaces, including cylinders and dead volumes, heater, cooler, and regenerator. This reduces the problem to a set of ordinary differential equations with respect to time, for pressure, velocity, and temperature, or entropy at the interfaces between the different spaces. We solve these equations numerically. Results are presented, which show that for typical cryocooler designs, losses due to irreversible heat transfer can be limited to a small fraction of the adiabatic loss. In contrast, the adiabatic loss remains roughly constant for all geometric designs with the same compression ratio.

Nomenclature

c_p = specific heat at constant pressure
c_v = specific heat at constant volume
d = tube diameter
f = arbitrary dimensionless factor
k = thermal conductivity
L = length
Nu = Nusselt number (based upon diameter)
Pr = Prandtl number
p = pressure
R = gas constant, $c_p - c_v$
Re = Reynolds number
St = Stanton number
s = entropy
T = temperature
t = time
U = velocity scale
u = velocity
x = position
γ = ratio of specific heats, c_p/c_v
ρ = density
τ = period

Subscripts
i = sequential time step number
L = corresponding to the left piston position
m = characterizes the regenerator matrix temperature
R = corresponding to the right piston position
w = characterizes the heat exchanger wall temperature
1 = corresponding to the interface left cylinder volume/heat exchanger volume
2 = corresponding to the interface heat exchanger volume/right cylinder volume

Superscripts
$+$ = for a discontinuous variable, value on the right side of a discontinuity
$-$ = for a discontinuous variable, value on the left side of a discontinuity
$*$ = initial time on a characteristic

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Introduction

OUR approach to modeling is predicated upon an uncompromising mathematical and numerical rigor, applied to a physical model that is simple enough so that a rigorous solution is feasible. Validation of the program is then completely independent of any physical issue; it consists of answering the following question: is the solution that is found, a true, valid approximation to the exact solution of the mathematical problem that is being posed? If that is the case, its validity having been established based upon its own merits, the numerical technique is presumably no longer an issue when discussing the results. It is then possible to analyze the results in light of the physical model exclusively. Therefore, the effect of simplifications of varying accurateness or realism that the physical model incorporates is not clouded by doubts that might arise regarding the numerical technique.

Thus, we consider a limited physical model that offers specific answers to a limited problem, but with a high degree of confidence. This was the case of the isothermal/isentropic model for adiabatic losses in Stirling cryocoolers,[1] which implemented a simple physical model, taking the heat exchangers as isothermal, and assumed that in the near-adiabatic cylinder spaces, the flow is isentropic and stratified,[1–3] i.e., time-dependent and one-dimensional in space. That model did not attempt to predict anything beyond one specific loss, the so-called adiabatic loss. That loss occurs because of the presence of the interface between near-adiabatic and near-isothermal regions in the device. Indeed, the isentropic and isothermal processes are both reversible, so that in the model, no irreversibility can occur within either volume. However, the temperature of the fluid delivered by the isentropic cylinder space to the neighboring isothermal heat exchanger normally does not match the heat exchanger temperature, so that an irreversibility is unavoidable at the interface. While that loss is independent of any finite rate mechanism, it is not predicted by the classical Schmidt model,[4] and it is not very amenable to empirical models.

In contrast with limited models that focus on specific losses, global numerical models attempt to provide global answers that take into account all significant aspects. But because of the difficulties they involve, and because of the magnitude of the numerical problem that results, their solution may require so many short cuts and compromises that they may turn out to be unreliable, complicated, or expensive. Global models[5-12] typically implement a numerical, usually finite difference, approximation of the conservation laws. Besides often being unwieldy, they also are inherently unable to divide the losses into contributions that can clearly be allocated to specific physical mechanisms.

Since our stratified model[1] remained a purely thermodynamic model in the sense that all results were independent of rate processes, predicted performance remained independent of the speed of the device. The heat exchangers were assumed to be isothermal, which is clearly not very realistic, except in some cases for the regenerator.

In this article, we propose a realistic heat exchanger model that can easily be coupled with the stratified cylinder model, but accounts fairly accurately for nonideal heat transfer in the freezer and cooler. For the regenerator, we retain the isothermal assumption. The new heat exchanger model shares one key property with the stratified cylinder model. All three models, for the cylinders, the heat exchangers, and the regenerator, lead to field equations that can be integrated in closed form with respect to the space variable, so that the global problem can be reduced to a set of ordinary differential equations that depend upon time only. In contrast with our previous model,[1] we now take into account that heat transfer is a time-dependent process, and performance results are thus no longer independent of the speed of the device.

The heat exchanger model is valid in the limit of a residence time in the heat exchanger that is negligible compared to the period of revolution of the machine. In that case, both flow and heat transfer can be analyzed as being quasisteady. The time-derivative terms in the conservation equations become negligible compared with convective terms, and the mass flow rate can be approximated as spatially uniform in the heat exchangers. Time now enters the equations only as a parameter, through the boundary conditions, which retain their time dependency. The residence time in the heat exchanger is small if the heat exchanger volume is small compared with the displacement.

Physical Model

In the same way as in the adiabatic model,[1] the device is represented as spatially one dimensional, and the cycle fluid is an ideal gas with constant specific heats at least locally. The cryocooler includes five main volumes: two cylinder spaces, the regenerator, the freezer, and the cooler. The heat exchangers consist of batteries of identical tubes, placed at the interfaces between each cylinder space and the regenerator, and their total volume is small compared with the displacements. As before, in the cylinders and dead volumes between cylinders and heat exchangers, the flow is isentropic and stratified, and the regenerator is isothermal.

The longitudinal x coordinate is rescaled so as to represent volume rather than length, but Reynolds and Nusselt numbers are calculated based upon actual areas, velocities, and sizes of the flow passages. The problem geometry is then as represented in Fig. 1. The spatial domain is the interval $[x_L(t), x_R(t)]$. $x_L(t)$ and $x_R(t)$ describe the piston positions (or swept

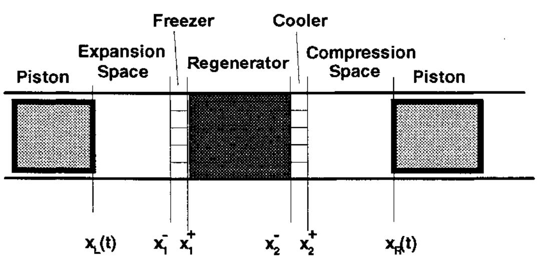

Fig. 1 Notation and indices.

volumes); they are known, periodic, functions of time. The expansion space, including adiabatic dead volume between cylinder and freezer, is the volume between x_L and x_1, and the compression space and dead volume is the volume between x_2 and x_R. Although x_L and x_R vary in time, x_1^-, x_1^+, x_2^-, and x_2^+, which represent the limits of the heat exchangers, are fixed. The regenerator is the volume between x_1^+ and x_2^-.

We assume pressure to remain spatially uniform. This is a leading-order approximation to the momentum equation that subsequently plays no further role in the model. Equations (1) and (2) are then the one-dimensional mass and energy conservation equations, with the pressure gradient term, which is negligible, dropped from the energy equation:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} \rho u = 0 \tag{1}$$

$$\rho c_p \frac{\partial T}{\partial t} + \rho c_p u \frac{\partial T}{\partial x} - \frac{\mathrm{d}p}{\mathrm{d}t} = \frac{4Nu(Re)k}{\mathrm{d}L} (T_w - T) \tag{2}$$

In Eq. (2), heat transfer has already been written in terms of an empirical convection model. There is no alternative to this, because the one-dimensional model is inherently unable to handle the true, multidimensional, representation of convection as a combination of conduction within the fluid, plus motion of the fluid. The Nusselt and Reynolds numbers are based upon the transverse scale d, which represents the size of the flow passages.

In the regenerator, heat transfer is very effective; so that the right side of Eq. (2) is then very large compared to the left side, and conservation of energy is reduced to the statement that the fluid temperature equals the temperature of the matrix.[13] Furthermore, if the thermal mass of the matrix is also very large, its temperature varies very little. If both the thermal mass and the heat transfer coefficient are infinite, the regenerator is isothermal: at any location within the regenerator, gas and matrix are at the same time-independent temperature, which is then determined by its (arbitrary) initial value. (To find the actual temperature profile attained in the periodic operation of an actual machine, the finite heat transfer rate and the finite thermal mass must be taken into account, so that the temperatures are allowed to float and eventually reach a stationary regime.[13]) However, our isothermal regenerator is reversible; there are no regenerator losses, and the only effect of the specific temperature profile adopted in our model is to determine the mass of fluid contained within the regenerator. We assume a linear profile, which actual distributions do not depart much from, and the resulting error will thus not be significant, compared with the loss that we neglect.

In the adiabatic spaces, a situation opposite to that in the regenerator occurs; the heat transfer area is small so that little heat transfer occurs. The right side of Eq. (2) is then much smaller than the left side. Taking into account that the spatial pressure gradient is zero, Eq. (2) with a zero right side implies conservation of entropy.

Finally, in the heat exchangers, an intermediate situation occurs; both sides of Eq. (2) play a role, and a more subtle simplifying assumption is called for: the quasisteady heat exchanger approximation.

The simplified heat exchanger model that we now propose is valid for small residence time, i.e., for $(L/U) \ll \tau$, in which L is the heat exchanger length, U is the magnitude of the velocity in the heat exchanger, and τ is the period. If $(L/U) \ll \tau$, then $(\partial/\partial t) \ll u(\partial/\partial x)$ so that time-dependent terms in Eqs. (1) and (2) are negligible compared to the convective terms $u(\partial/\partial x)$. In the continuity equation, Eq. (1), that hypothesis implies that the mass flow rate ρu is spatially uniform. In Eq. (2), $(\partial/\partial t) \ll u(\partial/\partial x)$ implies that the time-derivative of temperature is negligible compared to the convective term. Because $p = \rho RT$, and c_p/R is of order unity, the third term in Eq. (2), i.e., the time-derivative of pressure, has the same magnitude as the first term and it is thus also negligible.

Because (in our machine with unit cross section), the magnitude of the velocities equals the displacement divided by the period, the residence time in the cylinders is necessarily of order unity if the period is taken as the yardstick for time. Thus, the residence times in the heat exchangers will be negligible, provided that the volume of the heat exchangers is negligible compared with the displacements. The exchangers can consequently be represented as interfaces with zero length, so that $x_1^- \simeq x_1^+ \simeq x_1$, and $x_2^- \simeq x_2^+ \simeq x_2$. However, across these interfaces, temperature experiences a jump determined by the quasisteady approximation to Eq. (2), obtained by neglecting the time-dependent terms:

$$\rho c_p u \frac{\partial T}{\partial x} = \frac{4Nuk}{dL}(T_w - T) \tag{3}$$

Or, rearranging

$$\frac{\partial T}{\partial x} = \pm\frac{4Nu(Re)}{RePrL}(T_w - T) = \pm\frac{4St(Re)}{L}(T_w - T) \tag{4}$$

The $\pm$ sign on the right side of Eq. (4) is necessary because, although actual velocities can be in either direction, the Reynolds number is based upon the (unsigned) magnitude of velocity. The + sign corresponds to a positive velocity and vice versa. The Reynolds number depends upon ρu, which depends upon time, but is independent of x. Furthermore, if we neglect the fact that viscosity depends upon temperature that varies with x, then the coefficients on both sides of Eq. (4) are independent of x. Integration of Eq. (4) from x^- to $x^+ = x^- + L$ yields

$$T(x^+) = T_w + [T(x^-) - T_w]\exp[\mp 4St(Re)] \tag{5}$$

in which the $-$ sign is valid when velocity is positive and vice versa. The empirical relationship between the Stanton and Reynolds numbers must be obtained from experiments. Results below were obtained with the empirical correlation of Eq. (6), but if another formulation is more appropriate, it is a simple matter to insert it in the code:

$$Nu = 0.021Re^{0.8} \tag{6}$$

When the fluid in the heat exchanger is moving in the direction toward the cylinder, its temperature when entering the heat exchanger from the regenerator is the temperature of the regenerator end. While unknown initially, that temperature is time-independent, if the regenerator is isothermal. Assuming a known initial value for that temperature, Eq. (5) can directly be used to determine the temperature of the fluid entering the cylinder, after it has moved through the heat exchanger.

In the other direction, when the flow goes from the cylinder through the heat exchanger into the regenerator, Eq. (5) determines the temperature at which the fluid is delivered to the regenerator. That temperature varies with time. Normally, it will not match the fixed temperature of the regenerator matrix. A situation similar was encountered in the adiabatic-isothermal model.[1] In that model, the temperature of the extremity of the heat exchanger set was known, equal to the known heat exchanger temperature. But in that model the actual heat exchanger was part of the isothermal set, and an energy exchange with the outside world occurred in order to maintain the specified temperature. Now, we have only the regenerator proper within our isothermal volume; the regenerator is not in thermal contact with outside; instead, in the periodic regime it conserves energy, so that the energy brought in by the fluid entering from the heat exchanger is necessarily equal to the energy removed with the fluid when, later within the period, the fluid returns to the heat exchanger.

While Eq. (5) determines the temperature of the fluid entering the regenerator from the heat exchanger, after the velocity reverses, the fluid leaving the regenerator and moving into the heat exchanger has a temperature equal to the temperature of the regenerator matrix at its end. The net exchange between the fluid and the regenerator matrix over one full period is zero. The net enthalpy brought into the regenerator by the fluid, at the temperature determined by Eq. (5), when the flow goes into the regenerator, is necessarily equal to the enthalpy brought back to the heat exchanger when the flow returns, and is at the temperature of the extremity of the matrix. This determines the initially unknown temperature of the regenerator end.

The above set of approximations determines our physical model. Its mathematical equivalent is the following periodic boundary value problem characterized by Eq. (7).

The problem is to find time-periodic solutions $u(x, t)$, $p(t)$, $\rho(x, t)$, $T(x, t)$ and $s(x, t)$, with x defined in the interval $[x_L(t), x_R(t)]$, satisfying Eq. (7):

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\rho u = 0 \tag{7a}$$

$$T = T_m(x) \quad \text{for} \quad x \in (x_1, x_2) \tag{7b}$$

$$\frac{\partial s}{\partial t} + u\frac{\partial s}{\partial x} = 0 \quad \text{for} \quad x \notin (x_1, x_2) \tag{7b'}$$

$$p = \rho RT \tag{7c}$$

$$R\,ds = c_v\frac{dp}{p} - c_p\frac{d\rho}{\rho} \tag{7d}$$

Boundary conditions are

$$u(x_L) = \frac{dx_L}{dt} \tag{7e}$$

$$u(x_R) = \frac{dx_R}{dt} \tag{7f}$$

The jump condition at x_1 and x_2 is

$$(\rho u)^- = (\rho u)^+ \tag{7g}$$

$$T(x^+) = T_w + [T(x^-) - T_w]\exp\{\mp 4St[Re(\rho u)]\} \tag{7h}$$

$x_L(t)$ and $x_R(t)$ are the piston positions, which are known, periodic functions of time with period τ. T_w is equal to the freezer or cooler wall temperatures, which are known. $T_m(x)$ is an unknown linear function of x, such that the net total heat exchange with the regenerator, at each end, is zero. St is a known function of the Re, which is linear in ρu.

Equation (7a) is the continuity equation, Eq. (1). The regenerator is isothermal, hence, Eq. (7b). In the cylinders, material particles retain their entropy, hence, Eq. (7b'). Equations (7c) and (7d) are the equations of state. At the piston faces, the velocity of the fluid is equal to the piston velocity, providing us with the boundary conditions, Eqs. (7e) and (7f). Equation (7g) states that the mass flow rate is continuous in the (small) heat exchangers. Finally, Eq. (7h) is Eq. (5), the energy equation for the freezer and the cooler.

The same technical difficulties related to being well-posed arise as in the adiabatic model[1]; indeed, in the periodic boundary value problem as posed, the temperature of the fluid that never leaves the cylinder spaces is indeterminate. However, provided that the mean pressure is specified, results such as refrigeration and efficiency can be found, which do not depend upon these temperatures. Like before, these difficulties apply to the periodic boundary value problem, but not to the equivalent initial-boundary value problem that our numerical solution actually seeks by marching in time, because in the initial value problem, these temperatures are determined by their initial values.

Finally, inspection of the problem defined by Eq. (7) reveals that it only depends upon mean pressure and the period of revolution in the combination p/τ. Indeed, if all temperatures and entropies are maintained constant, but p and time are multiplied by an arbitrary factor f, Eqs. (7a), (7b), (7d), (7g), and (7h) remain unchanged, while in Eq. (7c), both sides are multiplied by f, and in Eqs. (7e) and (7f), both sides are divided by f.

Solution Technique

Our goal here is primarily to describe the mathematical process of solving the boundary value problem determined by Eq. (7). While, in the description that follows, we refer to the variables, equations, and coordinates by their physical names, all manipulations are valid and can be justified on their own merit without any physical reference.

The solution technique is similar to the solution that was adopted for the adiabatic model,[1] to which we have added the heat exchanger model. By replacing all other variables in the continuity equations as functions of pressure and its time derivatives, we obtain an equation that, after discretization, leads to an explicit, first-order accurate scheme. A periodic solution is then sought iteratively, by marching in time, starting from arbitrary initial conditions. Besides $p(t)$, the absolute constants $T_m(x_1)$ and $T_m(x_2)$ are also unknown; we start from an initial guess for all three unknowns. $p(t)$ is recalculated at each time step, whereas $T_m(x_1)$ and $T_m(x_2)$ are re-evaluated only once after each complete period. The iterative procedure is structured around Eq. (8), which is obtained by replacing pressure in Eq. (7a) by its value given by Eq. (7c) and integrating the resulting equation over the regenerator length, i.e., between x_1 and x_2:

$$\frac{\mathrm{d}p}{R\,\mathrm{d}t}\int_{x_1}^{x_2}\frac{1}{T_m}\,\mathrm{d}x + (\rho u)_2 - (\rho u)_1 = 0 \tag{8}$$

Since T_m is linear in x, the integral in the first term of Eq. (8) can be calculated from the guessed values of $T_m(x_1)$ and $T_m(x_2)$. We shall see also that if $T_m(x_1)$ and $T_m(x_2)$ are known, the values of the second and third terms in Eq. (8), $(\rho u)_1$ and $(\rho u)_2$, can be determined at each time step as functions of $p(t)$ and $\mathrm{d}p/\mathrm{d}t$, by solving for the cylinder and heat exchanger processes, therefore allowing for our iterative solution.

But we focus first on $T_m(x_1)$ and $T_m(x_2)$, which while time-independent, are unknown (see Physical Model section). Their value can actually be determined from the requirement that the net heat exchange with the regenerator is zero at each end:

$$\int_0^\tau \rho u T\,\mathrm{d}t = 0 \tag{9}$$

When the flow is in the direction from the regenerator into the freezer, the temperature of the fluid leaving the regenerator equals $T_m(x_1)$, the temperature at the extremity of the regenerator. When the flow is in the other direction the temperature of the fluid varies and depends upon the cylinder and heat exchanger processes. Splitting the integral in Eq. (9) by the direction of the flow, we have

$$\int_{\rho u>0}\rho u T\,\mathrm{d}t + T_m(x_1)\int_{\rho u<0}\rho u\,\mathrm{d}t = 0 \tag{10}$$

In our iterative solution, we start from a guessed value for $T_m(x_1)$, as we march in time we use Eq. (10) during each revolution (or period τ) to recalculate a new value of $T_m(x_1)$, that we then use for the next revolution. This requires us to accumulate the values of the two integrals: 1) the energy flow and 2) the mass flow into the regenerator, during that part of τ where the flow is in the direction into the regenerator, noting that at the periodic regime that is sought, the total mass flow rate in one direction becomes equal to the mass flow rate in the other direction. (Accumulation of the first integral requires that the temperature T of the fluid entering be known. At each time step, T will be determined by the heat exchanger and cylinder flow, as seen in the next paragraphs.) The cooler end of the regenerator is dealt with in the same way as the freezer. This provides us with updated values for $T_m(x_1)$ and $T_m(x_2)$ after each period, which eventually converge toward a steady, fixed value.

Next, to evaluate the terms $(\rho u)_1$ and $(\rho u)_2$ in Eq. (8), we consider the cylinders and heat exchangers. In Eq. (7b′), we replace $\mathrm{d}s$ as a function of pressure and density from Eq. (7d). Since pressure depends upon time only, its spatial derivative vanishes. Density is then eliminated, using its value from Eq. (7a), resulting in Eq. (11). Finally, the integration of Eq. (11) between x_L and x_1 leads to Eq. (12):

$$\frac{1}{\gamma p}\frac{\mathrm{d}p}{\mathrm{d}t} + \frac{\partial u}{\partial x} = 0 \tag{11}$$

$$u = \frac{\mathrm{d}x_L}{\mathrm{d}t} - \frac{1}{\gamma p}\frac{\mathrm{d}p}{\mathrm{d}t}(x - x_L) \tag{12}$$

According to Eq. (7g), ρu is constant throughout the heat exchangers. Thus, its value on the regenerator side of the heat exchanger in Eq. (8) can be replaced by its value on the cylinder side. Velocity on the cylinder side is given by Eq. (12). To find the density on the cylinder side as a function of pressure, we still need the temperature of the fluid entering or leaving the cylinder. Fluid entering the cylinder has a temperature determined by the heat exchanger, it is given by Eq. (7h), knowing that the fluid that entered the heat exchanger, at its other extremity on the regenerator side, was at $T_m(x_1)$, for which we have a value.

When the flow is in the other direction, the temperature of the fluid delivered by the cylinder to the heat exchanger depends upon the history of the flow in the cylinder. While in the cylinder, each parcel of fluid retains its entropy, so that the entropy of the parcel of fluid returning to the heat exchanger is equal to the entropy of that same parcel when it entered the cylinder. Thus, we need to track the path of each particle in the cylinder, or equivalently, we need to solve the entropy equation, Eq. (7b′), on characteristics defined by Eq. (13). On those characteristics, entropy is constant; Eq. (7b′) becomes $\mathrm{d}s = 0$. (However, different characteristics carry different values of entropy.)

$$u(x, t) = \frac{\mathrm{d}x}{\mathrm{d}t} \tag{13}$$

Replacing u in Eq. (13) by its value from Eq. (12), and integrating between t^*, at which a parcel of fluid leaves x_1, and t, when that parcel returns to x_1, we obtain Eq. (14):

$$\frac{p(t)}{p(t^*)}\left[\frac{x - x_L(t)}{x^* - x_L(t^*)}\right]^\gamma = 1 \tag{14}$$

Knowing the past pressure history, Eq. (14) allows us to find the time t^* at which the parcel that returns to the interface cylinder-heat exchanger at t had previously left that interface.

Since the entropy of that parcel of fluid remained constant during that process, and knowing that when the particle left x_1, its temperature was equal to the (known) temperature of the fluid delivered by the heat exchanger, Eq. (14) can be used to determine the temperature of the fluid returning to the interface at t.

Finally, Eq. (7h) determines the temperature of the fluid exiting the heat exchanger into the regenerator. As we saw above, that temperature is needed to evaluate the first term in Eq. (10), which is used, after one full period, to recompute a new value of T_m at the regenerator end. After flow reversal, the fluid delivered to the heat exchanger from the regenerator is then at the fixed temperature T_m.

The information necessary to construct an iterative solution for pressure is thus complete. However, Eq. (14) determines only implicitly the relationship between t and t^*, so that at each time step an iterative search is required. In principle, our initial set of hyperbolic partial differential equations has been reduced to an ordinary differential equation for pressure, albeit with a time shift of magnitude that varies over the cycle and that can only be determined by a search process over the previous history of the cycle. Replacing $(\rho u)_1$ and $(\rho u)_2$ in Eq. (8) by the values of u from Eq. (12), and expressing ρ from the gas law, we obtain Eq. (15):

$$\frac{\mathrm{d}p}{R\,\mathrm{d}t}\int_{x_1}^{x_2}\frac{1}{T_m}\,\mathrm{d}x + \frac{p}{RT_2^+}\left[\frac{\mathrm{d}x_R}{\mathrm{d}t} - \frac{1}{\gamma p}\frac{\mathrm{d}p}{\mathrm{d}t}(x_2 - x_R)\right] - \frac{p}{RT_1^-}\left[\frac{\mathrm{d}x}{\mathrm{d}t} - \frac{1}{\gamma p}\frac{\mathrm{d}p}{\mathrm{d}t}(x_1 - x_L)\right] = 0 \tag{15}$$

Finally, the temperatures T_2^+ and T_1^- can be replaced by their values calculated following either of the two procedures described above, depending upon the direction of the flow.

For details and a complete discussion about the numerical implementation, we refer to our adiabatic, stratified model.[1] The main features of the scheme can be summarized as follows. A periodic solution is reached by marching in time. The scheme is only first-order accurate because the iterative search on the pressure history is essentially first-order accurate. Pressure is renormalized after each revolution to compensate for the numerical error on mass conservation, since the scheme is not inherently mass-conservative. As before, a time step of $\frac{1}{32}$ of one crank angle degree results in a numerical error that, compared with results at significantly finer resolutions, is found to be insignificant.

Results

We used the code described above (the source code is available for inspection and noncommercial use as per the conditions specified in the program copyright notice), to model a cryocooler with a design somewhat similar to the Philips cryocooler with magnetic bearings.[14,15] Helium was adopted as the cycle fluid. The temperatures were 65 and 285 K, respectively, for the freezer and cooler walls. Dimensions were as follows, using the regenerator volume as the unit volume and its net cross section as the unit cross section: 1) expansion space: displacement = 0.166, dead volume = 0.112; 2) compression space: displacement = 1.555, dead volume = 2.250; 3) regenerator: length = 63 mm; 4) freezer: total net cross section = 0.072; tube diameter = 0.2 mm; length = 44 mm; 5) cooler: total net cross section = 0.162; tube diameter = 0.7 mm; length = 63 mm; and 6) phase angle (between piston faces): varied between 18–90 deg.

As mentioned above, our physical model yields the same solution for all combinations of pressure and frequency with the same product, which we varied in the range from 1.62 MPa × rpm to 1.62 10^6 MPa × rpm. The lower value corresponds to a machine that is so slow that the effects of finite rate mechanisms become negligible, with results comparable with those from the isentropic/isothermal model.[1] The upper limit was high enough so that a substantial performance drop could be observed.

Results are shown in Figs. 2–5. As speed approaches zero, heat transfer rates approach infinity and the only loss that remains is the adiabatic loss. We verified that our results approach, as they should, the results given by the adiabatic model with ideal, isothermal heat exchangers. That limit-case corresponds to the lower range of abscissae, for which the results are indeed undistinguishable from those given by the adiabatic model. Figure 2 shows that the adiabatic loss is larger at larger phase angles. However, the refrigeration predicted by the adiabatic model is also larger, so that, on Fig. 3, while the relative refrigeration decreases as the phase angle is increased, its actual value increases considerably, starting from zero for a zero phase angle, and reaching a maximum for a phase angle just under 90 deg.

In Fig. 2, we see that for a moderate pressure × speed combination, corresponding roughly to the design conditions for the Philips cryocooler, i.e., a pressure in the 1–2-MPa range and speeds in the 1000–3000-rpm range, the effect of nonideal heat transfer in the freezer and cooler is minimal compared with the adiabatic loss, which is substantial and

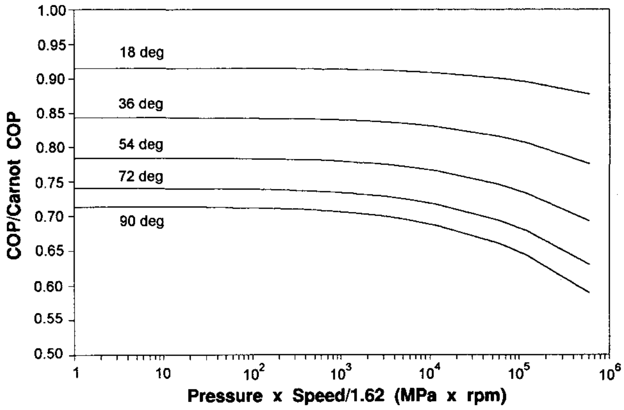

Fig. 2 COP/Carnot COP vs speed, at various phase angles.

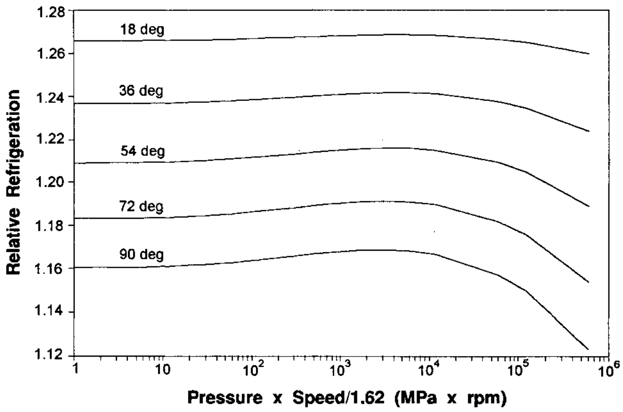

Fig. 3 Refrigeration relative to isothermal, vs speed, at various phase angles.

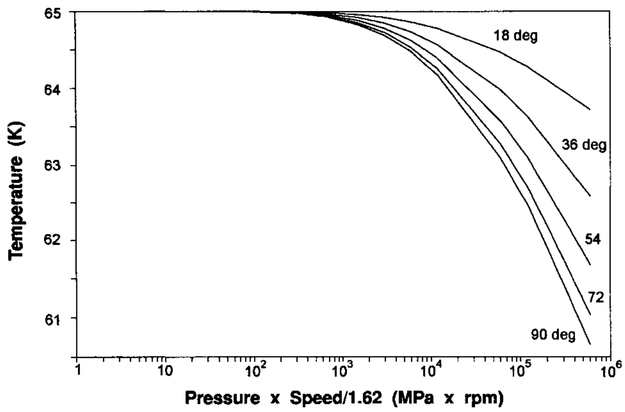

Fig. 4 Temperature at cold end of regenerator, vs speed, at various phase angles.

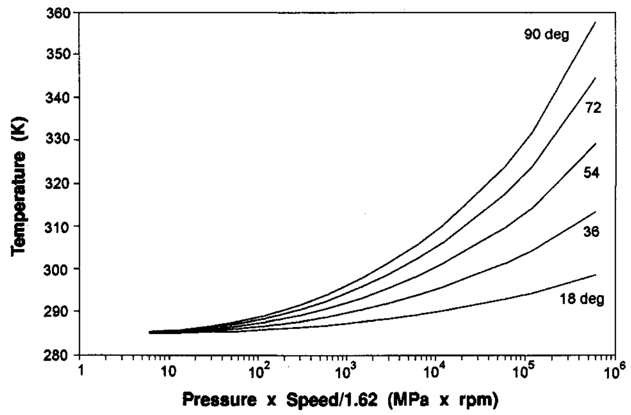

Fig. 5 Temperature at warm end of regenerator, vs speed, at various phase angles.

cannot be reduced without also reducing the specific refrigeration. This happens even though, as Figs. 4 and 5 show, in that range of pressure $\times$ velocity, the temperatures at the two ends of the regenerator are already significantly different from the heat exchanger wall temperature, especially on the cooler side.

Figure 3 presents the ratio refrigeration relative to isothermal refrigeration. Isothermal refrigeration, i.e., refrigeration as predicted by a Schmidt analysis, was chosen as the reference because, besides being proportional to pressure and speed, it also accounts for most of the effect of the phase angle on the compression ratio. (But adiabatic cylinders result in a higher compression ratio than isothermal cylinders, hence, a relative refrigeration greater than one.)

In Fig. 3, the ratio refrigeration relative to isothermal refrigeration is shown to increase slightly with speed or pressure, before eventually deteriorating at high speed $\times$ pressure combinations. However, Fig. 2 shows that the slight increase in refrigeration is accompanied by a slight decrease in efficiency. Refrigeration is affected by factors such as the phase angles between pressure and volume in the cylinders, which depend upon the temperature at which the fluid is delivered from the heat exchangers into the cylinders. Nonideal heat transfer in the heat exchangers has an effect on the phase angles between pressure and cylinder volume, which in the case that we modeled, resulted in a small increase in refrigeration, and may actually have had a positive effect on efficiency. That may explain why efficiency deteriorates so little with frequency in Fig. 2.

One cannot, however, conclude from our results that heat exchanger losses are always insignificant and would therefore not be worth analyzing. This model will remain useful, even though the results that we present show that the particular loss that we address is not necessarily significant. Indeed, for nonoptimal heat exchangers, the loss will be large.

Furthermore, our physical model includes only one of the various time-dependent rate mechanisms that affect performance. As our results show, it may in some cases be possible to design heat exchangers with minimal losses, and in that case, other rate mechanisms such as viscous stresses, mechanical losses, regenerator ineffectiveness, and fluid leakage across piston seals will then likely have a much more dramatic impact on overall performance. Since we do not account for those potentially much more significant mechanisms, the current model cannot be expected to predict the global effects of pressure and speed, except perhaps in a narrow range. The model will likely not predict trends observed in experiments in which pressure and speed are varied. The use of the model is thus limited to the prediction of adiabatic and heat losses exchanger. Other losses, such as conduction losses, or mechanical losses, depend mostly on a different set of design parameters, and it will often be more effective to evaluate them separately.

Conclusions

The proposed model can be used very effectively to study or predict adiabatic and heat exchanger losses. For well-designed heat exchangers, the model shows that in some situations, such as the one we modeled, it is possible to reduce the heat exchanger loss to an insignificant fraction of the adiabatic loss. The adiabatic loss is due to the cylinders delivering fluid to the heat exchangers at the wrong temperature. Thus, the actual irreversibility that the adiabatic loss entails occurs as an entry effect in the heat exchangers, which is unavoidable regardless of the effectiveness of the heat exchanger, and, except for the compression ratio, is relatively insensitive to the geometric design. If besides the adiabatic cylinders, nonideal heat transfer is taken in consideration, the results change in three different ways. There is an additional irreversibility in the heat exchangers. But also, the temperature of the fluid delivered to the cylinders is modified, which indirectly affects the magnitude of the adiabatic loss. Finally, in the adiabatic model, the adiabatic loss physically appeared as an irreversible discontinuous temperature jump, associated with a finite, discrete amount of heat transfer to the wall at the heat exchanger entry contiguous to the cylinder. With the heat exchanger model added, that jump is now resolved and the spatial distribution of the corresponding heat exchange is now accounted for. Our results indicate that under appropriate conditions, the presence of nonideal heat exchangers may actually slightly reduce the adiabatic loss, at least to the extent that the total loss does not significantly increase in the presence of irreversible heat transfer.

Our model is based upon a closed-form integration with respect to the space variable. Only in time is a numerical solution required. This leads to a model that is very efficient and cheap to implement numerically. As a result, the program is very quick, which makes this model suitable for systematic investigations of large portions of the parameter space. It is a good alternative to the isothermal models often used as the core of simple Stirling models, because the adiabatic loss is evaluated by modeling the fundamental processes rather than by an empirical model. However, if realistic overall performance results are to be predicted with any degree of accuracy, all other significant losses that this model is not concerned with must be evaluated and accounted for.

References

[1]Bauwens, L., "Adiabatic Losses in Stirling Cryocoolers—A Stratified Flow Model," *Cryogenics*, Vol. 34, No. 8, 1994, pp. 627–633.

[2]Smith, J. L., Jr., and Romm, M. J., "Thermodynamic Loss at Component Interfaces in Stirling Cycles," *Proceedings of the 27th Intersociety Energy Conversion Engineering Conference*, Vol. 5, Society of Automotive Engineers, Warrendale, PA, 1992, pp. 529–532.

[3]Romm, M. J., and Smith, J. L., Jr., "Stirling Engine Loss in Dead Volumes Between Components," *Proceedings of the 28th Intersociety Energy Conversion Engineering Conference*, Vol. 2, American Chemical Society, Washington, DC, 1993, pp. 751–758.

[4]Urieli, I., and Berchowitz, D. M., *Stirling Engine Analysis*, Hilger, Bristol, England, UK, 1984.

[5]Gary, J., and Radebaugh, R., "A Numerical Model for Regenerator Performance [REGEN 3], Interim Report for Period May 1987–December 1988," Flight Dynamics Lab., WRDC-TR-89-3049, Wright-Patterson AFB, OH, 1989.

[6]Gedeon, D. R., "A Globally-Implicit Stirling Cycle Simulation," *Proceedings of the 21st Intersociety Energy Conversion Engineering Conference*, American Chemical Society, Washington, DC, 1986, pp. 550–556.

[7]Bauwens, L., and Mitchell, M. P., "Regenerator Analysis: Validation of the MS*2 Stirling Cycle Code," *Proceedings of the 18th International Congress of Refrigeration*, International Inst. of Refrigeration, Paris, 1991, pp. 930–934.

[8]Bauwens, L., "Stirling Engine Modeling: The MS*2 Code," *Proceedings of the International Stirling Engine Conference*, Rome, 1993, pp. 371–376 (Paper 081).

[9]Bauwens, L., "Stirling Engine Modeling on Eulerian Grids," *Proceedings of the 28th Intersociety Energy Conversion Engineering Con-*

ference, Vol. 2, American Chemical Society, Washington, DC, 1993, pp. 725–730.

[10]Organ, A. J., *Thermodynamics and Gas Dynamics of the Stirling Cycle Machine*, Cambridge Univ. Press, London, 1992.

[11]Bauwens, L., "Stability, Consistency, Convergence of Stirling Engine Models," *Proceedings of the 25th Intersociety Energy Conversion Engineering Conference*, American Institute of Chemical Engineers, New York, 1990, pp. 352–358.

[12]Bauwens, L., "Numerical Modeling of Stirling Cryocoolers," AIAA Paper 94-0130, Jan. 1994.

[13]Bauwens, L., "The Near-Isothermal Regenerator: A Perturbation Analysis," *Proceedings of the 29th Intersociety Energy Conversion Engineering Conference*, AIAA, Washington, DC, 1994.

[14]Stolfi, F., Goldowski, M., Keung, C., Knox, L., Lindale, E., Maresca, R., Ricciardelli, J., and Shapiro, P., "Design and Fabrication of a Long-Life Stirling Cycle Cooler for Space Applications," Philips Labs., Briarcliff Manor, NY, March 1983.

[15]Stolfi, E., "Final Report for Parametric Testing of Engineering Model Refrigerator," Philips Labs. Rept. PL-8-CR84-0216, Briarcliff Manor, NY, Feb. 1984.